The Symbiotic Relationship of Combinatorics and Matrix Theory

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ABSTRACT

This article demonstrates the mutually beneficial relationship that exists between combinatorics and matrix theory.

1. INTRODUCTION

According to *The Random House College Dictionary* (revised edition, 1984) the word *symbiosis* is defined as follows:

symbiosis: the living together of two dissimilar organisms, esp. when this association is mutually beneficial.

In applying, as I am, this definition to the relationship between combinatorics and matrix theory, I would have preferred that the qualifying word "dissimilar" was omitted. Are combinatorics and matrix theory really dissimilar subjects? On the other hand I very much like the inclusion of the phrase "when the association is mutually beneficial" because I believe the development of each of these subjects shows that combinatorics and matrix theory have a mutually beneficial relationship. The fruits of this mutually beneficial relationship is the subject that I call *combinatorial matrix theory* or, for brevity here, CMT. Viewed broadly, as I do, CMT is an amazingly rich and diverse subject. If it is

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combinatorial and uses matrix theory[1] in its formulation or proof, it's CMT. If it is matrix theory and has a combinatorial component to it, it's CMT. Others may not view CMT in such broad terms. But in my mind it is the breadth and diversity of CMT that make it such a fascinating subject. The word "combinatorial" in CMT often means graph-theoretic, at least to those who call themselves matrix theorists. But there are a lot of other parts of combinatorics that fall within CMT, e.g. a good portion of the theory of designs.[2]

In the remainder of this section I would like to address the question:

Is it surprising that there should be a mutually beneficial relationship between combinatorics and matrix theory?

and at the same time to address the not unrelated question already posed above:

Are combinatorics and matrix theory really dissimilar?

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an m-by-n matrix with, say, entries from the complex number field $\mathscr{C}$. Denote the set of the first k positive integers by $\mathscr{X}_k$. Then A may be regarded as a function

$$f = f_A : \mathscr{X}_m \times \mathscr{X}_n \to \mathscr{C}.$$

The *kernel* of f is

$$\ker f = \{(i, j) : f(i, j) = 0\}.$$

We may regard $\ker f$ as a bipartite graph as follows. Let $K_{m,n}$ denote the complete bipartite graph with vertex bipartition $\mathscr{X}_m$ and $\mathscr{X}_n$. In $K_{m,n}$ each vertex i of $\mathscr{X}_m$ is joined by an edge (i, j) to each vertex j of $\mathscr{X}_n$.[3] The

[1] I include under the heading *matrix theory* the various parts of linear algebra which have implications for matrices.

[2] Perhaps "combinatorial" ought to be replaced by "discrete," but *discrete matrix theory* seems misleading to me, although *discrete matrix analysis* seems to convey the right idea for a significant part of CMT.

[3] Even though by our definition one of $\mathscr{X}_m$ and $\mathscr{X}_n$ is a subset of the other, it should cause little confusion to regard them, as we do here, as disjoint sets.

elements of $\ker f$ are edges of $K_{m,n}$, and we may now think of $\ker f$ as the spanning[4] bipartite subgraph of $K_{m,n}$ with these edges. The *complementary* spanning bipartite graph

$$\overline{\ker}\, f = K_{m,n} - \ker f$$

has edges which locate the nonzero entries of the matrix A:

$$a_{ij} \neq 0 \text{ if and only if } (i, j) \text{ is an edge of } \overline{\ker}\, f.$$

Alternatively, we can view the complementary graph $\overline{\ker} f$ as a weighted bipartite graph: There is an edge (i, j) between vertex i and vertex j of *weight* a_{ij}. Thus $a_{ij} = 0$ signifies an edge of weight 0, that is, no edge between i and j. For example, let

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 0 & 1.2 & 0 & -1 \\ 5 & 0 & 3 & -6 \end{bmatrix}.$$

The weighted bipartite graph $\overline{\ker} f$ is shown in Figure 1. The numbers next to the edges are the weights of the edges. We call $\overline{\ker} f$ the (*weighted*) *bipartite graph* of A.

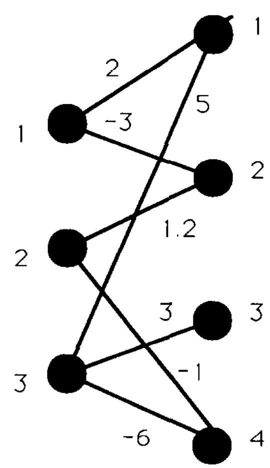

FIG. 1.

[4] *Spanning* means that this graph has the same set of vertices as $K_{m,n}$.

Now assume that $m = n$, so that A is a square matrix of order n. Let $\overleftrightarrow{K}_n$ denote the complete digraph of order n. The vertex set of $\overleftrightarrow{K}_n$ is the set $\mathscr{X}_n$ of the first n positive integers, and all ordered pairs (i, j) with i and j in $\mathscr{X}_n$ form *arcs*, including *loops* of the form (i, i). We may now view $\ker f$ and $\overline{\ker f}$ as spanning subdigraphs of $\overleftrightarrow{K}_n$. The arcs of $\ker f$ locate the zeros of A; the arcs of the complementary digraph $\overline{\ker f}$ locate the nonzeros of Z. We can also view $\overline{\ker f}$ as a weighted digraph in which the arc (i, j) has weight a_{ij} (again $a_{ij} = 0$ signifies an arc of weight 0 and thus no arc from i to j). The weighted digraph corresponding to the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 3 & 5 \\ -4 & 0 & 0 & 1 \\ 3 & 1 & -1 & 0 \end{bmatrix}$$

is drawn in Figure 2.

Now suppose that, in addition to $m = n$, we also assume that A is a symmetric matrix. Let K_n° denote the complete graph with vertex set $\mathscr{X}_n$ in which every pair $\{i, j\}$ of vertices i and j, including $i = j$, form an edge.[5] Then $\overline{\ker f}$ can be viewed as a weighted spanning subgraph of K_n°. This is

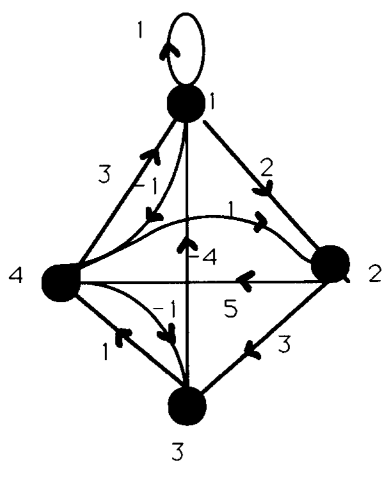

FIG. 2.

[5] The complete graph in which every pair of *distinct* vertices form an edge is denoted by K_n.

illustrated for the symmetric matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -3 & 2 \\ 2 & -3 & 0 & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

in Figure 3.

The application of the definition of *symbiosis* to CMT is now in doubt, since it refers to *dissimilar* organisms. It seems that a matrix *is* a weighted combinatorial object:

rectangular matrix	:	weighted bipartite graph
square matrix	:	weighted digraph
symmetric matrix	:	weighted graph.

The preceding observations do not yet explain *why* combinatorial ideas have had a significant impact on matrix theory and *why* matrix theory has had a significant impact on combinatorics.

What are the fundamental concepts of matrix theory?

It can be argued that there are two primary concepts in matrix theory on which most if not all of the other fundamental concepts depend. They are

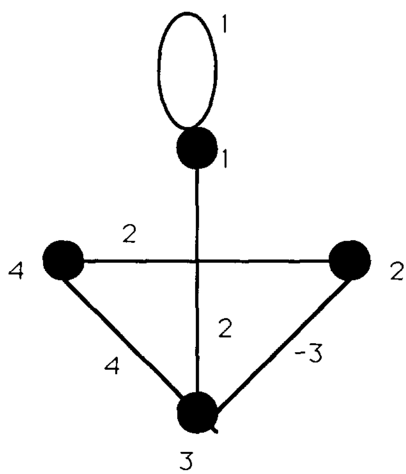

FIG. 3.

rank and *eigenvalue*. The dependence of some of the other basic concepts on rank and eigenvalue is indicated in the table below:

Rank	Eigenvalue
Nonsingularity and inverses	Positive definite
Solvability of $Ax = b$	Stability
Canonical forms:	Singular values, matrix norm
Row echelon form	Canonical forms:
LU factorization	Diagonal form
$\vdots$	Jordan form
	QR factorization
	$\vdots$

But both rank and eigenvalue can be defined in terms of the more basic concept of the *determinant* of a matrix:

the rank of a matrix A equals the largest order of a submatrix B of A with

$$\det B \neq 0;$$

An eigenvalue of a square matrix A is a root of the equation

$$\det(\lambda I - A) = 0.$$

Thus *the determinant is the source of a vast amount of matrix theory*. Having made that (rather obvious) observation, I can now at least partly explain why the association of combinatorics and matrix theory has been and is mutually beneficial, i.e. why the relationship between combinatorics and matrix theory is *symbiotic*.

Let $A = [a_{ij}]$ be a matrix of order n. Then the *determinant* of A is defined by

$$\det A = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where the summation extends over all permutations σ in the symmetric group S_n. It is possible to express the determinant of A in terms of the weighted bipartite graph we have associated with A and also in terms of the associated weighted digraph.

I. *Bipartite-graph formulation of the determinant.* To each permutation σ in S_n there corresponds the collection

$$F_\sigma = \{(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))\}$$

of edges of the complete bipartite graph $K_{n,n}$. The set F_σ consists of n pairwise vertex-disjoint edges which touch each vertex exactly once and is called a *1-factor* (or *perfect matching*) of $K_{n,n}$. There is a one-to-one correspondence between permutations in S_n and 1-factors of $K_{n,n}$. In the weighted bipartite graph of A each edge has a weight,[6] and we use these weights to define the weight of a 1-factor F_σ by

$$\text{wt } F_\sigma = (\text{sign } \sigma)\,(\text{product of the weights of the edges of } F_\sigma).$$

The weight of the set $\mathscr{F}_n$ of all 1-factors of $K_{n,n}$ (being the sum of the weights of the individual 1-factors) is thus equal to the determinant of A:

$$\det A = \text{wt } \mathscr{F}_n.$$

Therefore *the determinant is a function which gives the weight of the set of configurations in $K_{n,n}$ known as 1-factors, where the weights are determined by the matrix argument.*

II. *Digraph formulation of the determinant.* To each permutation σ in S_n there corresponds the same set F_σ, but now F_σ is a set of n arcs of the complete digraph $\overleftrightarrow{K}_n$. The set F_σ has the property that exactly one arc exits each vertex and exactly one arc enters each vertex. Such a set of n arcs is called a *1-factor of the digraph* $\overleftrightarrow{K}_n$.

There is a one-to-one correspondence between the permutations in S_n and the 1-factors of $\overleftrightarrow{K}_n$. In the weighted digraph associated with A each arc has a weight, and we use these weights to define the weight of a 1-factor F_σ as follows. A 1-factor F_σ is a spanning set of pairwise vertex disjoint directed cycles of $\overleftrightarrow{K}_n$; these directed cycles correspond to the usual decomposition of the permutation σ into disjoint (permutation) cycles. The weight of a directed

[6] Including the weight 0 for the non-edges.

cycle γ is defined by

$$\text{wt } \gamma = -\left(\text{product of the weights of the arcs of } \gamma\right),$$

and the weight of the 1-factor F_σ is defined by

$$\text{wt } F_\sigma = \text{product of the weights of the directed cycles of } F_\sigma .$$

Suppose that F_σ has $k \geqslant 1$ directed cycles. Then

$$(-1)^k = (-1)^n(-1)^{n-k} = (-1)^n \text{ sign } \sigma,$$

and hence

$$\text{wt } F_\sigma = (-1)^n(\text{sign } \sigma)\left(\text{product of the weights of the arcs of } F_\sigma\right).$$

Hence it follows that

$$\det A = (-1)^n \text{ wt } \mathscr{D}_n,$$

where $\mathscr{D}_n$ is the set of 1-factors of $\overleftrightarrow{K}_n$, and thus

$$\det(-A) = \text{wt } \mathscr{D}_n .$$

The above considerations certainly suggest that 1-factors of bipartite graphs and 1-factors of digraphs should have an impact on matrix properties. There is one important observation to be made regarding the use of bipartite graphs and digraphs in matrix theory. Some matrix properties do not change under arbitrary row and column permutations (or change but are directly recovered by applying the inverse permutations; we consider this as not changing); others do not change under simultaneous row and column permutations. In the former case it is usually the bipartite graph that has an impact on the matrix property, because except for the labeling of the vertices in each set of the bipartition, the bipartite graphs of a matrix A and the matrix PAQ (P and Q are permutation matrices) are the same. In the latter case it is the digraph that has an impact on the matrix property, because except for labeling of the vertices the digraphs of A and PAP^T are the same. This division of matrix properties into two types is in agreement with the division of matrix properties into two types given by the table. The rank of a matrix does not change under arbitrary permutations; the eigenvalues do not change under simultaneous permutations.

We close this introductory section with two simple examples which illustrate some of the above discussion.

EXAMPLE 1. Let A be a nonsingular matrix of order n. Then there exist permutation matrices P and Q of order n such that

$$PAQ = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix},$$

a triangular matrix, if and only if in the weighted bipartite graph associated with A there is exactly one 1-factor of nonzero weight.

Note that the conclusion does not hold if the hypothesis of nonsingularity is removed and we replace the words "exactly one" with "at most one."

EXAMPLE 2. Let

$$A = \begin{bmatrix} 0 & r_1 & 0 & \cdots & 0 \\ 0 & 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{n-1} \\ r_n & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The famous theorem of Geršgorin tells us that each eigenvalue λ of A satisfies

$$|\lambda| \leqslant \max\{r_1, r_2, \ldots, r_n\}.$$

But

$$\det(\lambda I - A) = \lambda^n + (-r_1 r_2 \cdots r_n),$$

where $-r_1 r_2 \cdots r_n$ is the weight of the unique 1-factor (with a nonzero weight) of the weighted digraph associated with A. (In this case the 1-factor is a directed cycle.) Thus each eigenvalue λ satisfies

$$|\lambda| = \sqrt[n]{r_1 r_2 \cdots r_n}.$$

In the sections that follow my goal is to highlight the mutually beneficial relationship that exists between combinatorics and matrix theory. In order to keep this article to a reasonable length and to keep it from being a chore to read, I have been in some instances selective in what I have chosen to discuss and brief in my discussions. My main purpose is to demonstrate the partnership that has existed between combinatorics and matrix theory, a partnership which is flourishing today and, in my opinion, will continue to flourish into the 21st century.

2. DIGRAPHS IN THE SPECTRAL THEORY OF MATRICES

It was Frobenius' great insight that certain spectral properties of a nonnegative matrix depend only on the locations of the nonzero elements of the matrix (and thus not on the magnitudes of these elements), that is, depend only on the (unweighted) digraph associated with the matrix. Generalizing work of Perron (1907) on positive matrices, Frobenius (1912) developed a theory of nonnegative matrices which is now known as the Perron-Frobenius theory of nonnegative matrices. This theory was further developed by Romanovsky (1936), Wielandt (1950), Pták (1958), Pták and Sedláček (1958), Varga (1962), and Dulmage and Mendelsohn (1967).

The property of *irreducibility*[7] of a matrix introduced by Frobenius [but for some historical perspective see Schneider (1977)] is now known to be equivalent to the associated digraph of the matrix being *strongly connected* (for all ordered pairs of vertices i and j there is a directed walk from i to j). Some of the main conclusions of the Perron-Frobenius theory for irreducible nonnegative matrices A are:

The largest modulus ρ of an eigenvalue[8] *is itself an eigenvalue and, as a root of the characteristic equation, is simple. There is a positive eigenvector corresponding to the eigenvalue ρ.*

The number k of eigenvalues with modulus equal to ρ equals the greatest common divisor of the lengths of the directed cycles in the associated digraph. In particular, there is a unique eigenvalue of maximum modulus ρ if and only if the lengths of the directed cycles are relatively prime.

[7] We shall refrain from defining most matrix terms and some graphical terms which we think will be familiar to most readers. Some terms will be defined in the footnotes.

[8] Called the *spectral radius* of a matrix.

If A is primitive, that is, $k = 1$, then there is a smallest positive integer p, called the exponent of A and denoted by $\exp A$, *such that A^p is a positive matrix. The exponent* $\exp A$ *is the smallest positive integer p such that for all ordered pairs of vertices i and j there is a directed walk from i to j of length exactly p.*

The above conclusions demonstrate quite clearly that the digraph of a nonnegative matrix has a strong influence on its spectral properties.

Kellogg and Stephens (1978) formulated the problem of determining the set $\Lambda_\rho(\Gamma)$ of all possible eigenvalues of the set $\mathcal{M}_\rho(\Gamma)$ of all nonnegative matrices of order n with a given spectral radius ρ whose associated digraph is a spanning subdigraph of a prescribed digraph Γ of order n.[9] If Γ is the complete digraph[10] $\overleftrightarrow{K}_n$, then this problem was solved by Dimitriev and Dynkin (1945, 1946) and Karpelevich (1951).

Let m be the length of the longest directed cycle of the digraph Γ of order n. Then Kellogg and Stephens proved:

If $m = 2$, then $\Lambda_\rho(\Gamma)$ consists only of real numbers.

If $m > 2$, then each eigenvalue $\lambda = \mu + i\nu$ in $\Lambda_\rho(\Gamma)$ satisfies

$$\mu + |\nu| \tan \frac{\pi}{m} \leqslant \rho.$$

They conjectured that if λ is in $\Lambda_\rho(\Gamma)$ for some Γ, then λ is in $\Lambda_\rho(\overleftrightarrow{K}_m)$. In words, eigenvalues of matrices whose digraphs have no directed cycles of length greater than m are eigenvalues of nonnegative matrices of order m.

The set $\Lambda_\rho(\Gamma)$ was determined by Johnson, Kellogg, and Stephens (1979) in case the digraph obtained from Γ by removing any loops is either a directed cycle or a directed cycle with a chord from a vertex to another vertex at distance 2. The subset of $\mathcal{M}_\rho(\Gamma)$ consisting of the matrices with trace zero was also considered.

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[9] Thus the set considered is the set of all nonnegative matrices with a given spectral radius ρ having zeros in a prescribed set of positions and possibly elsewhere. The existence of a positive eigenvector corresponding to the spectral radius ρ of an irreducible nonnegative matrix A implies that A is diagonally similar to a nonnegative matrix B all of whose row sums equal ρ. Thus B also has spectral radius ρ, indeed the same eigenvalues as A, and B has zeros in exactly those positions that A has zeros, that is, A and B have the same associated digraph. Thus it suffices to consider only the B's. Without loss of generality we could assume that $\rho = 1$, and thus consider only the row stochastic matrices with 0's in prescribed positions.

[10] Thus $\mathcal{M}_\rho(\Gamma)$ is the set of *all* nonnegative matrices of order n with spectral radius ρ.

Wielandt (1950) noted that the largest exponent of an irreducible nonnegative matrix of order n equals $n^2 - 2n + 2$. Let E_n be the set of numbers which can serve as the exponent of an irreducible nonnegative matrix of order n. Then

$$E_n \subseteq \{1, 2, \dots, n^2 - 2n + 2\}.$$

Dulmage and Mendelsohn (1964) showed that the above containment is proper for $n \geqslant 4$, that is, that there are *gaps* in the exponent set E_n.[11] Lewin and Vitek (1981) obtained a family of gaps in E_n and completely described the set

$$E_n \cap \left\{ \left\lfloor \frac{n^2 - 2n + 2}{2} \right\rfloor + 2, \dots, n^2 - 2n + 2 \right\}.$$

In addition they *conjectured* that

$$\left\{ 1, 2, \dots, \left\lfloor \frac{n^2 - 2n + 2}{2} \right\rfloor + 1 \right\} \subseteq E_n.$$

The following counterexample was found by Shao (1985):

$$n = 11 \quad \Rightarrow \quad \left\lfloor \frac{n^2 - 2n + 2}{2} \right\rfloor + 1 = 51, \quad \text{but} \quad 48 \notin E_{11}.$$

Shao proved that:

$\{1, 2, \dots, \lfloor (n^2 - 2n + 2)/4 \rfloor + 1\} \subseteq E_n$.

$\{1, 2, \dots, \lfloor (n^2 - 2n + 2)/2 \rfloor + 1\} \subseteq E_n$ *for n sufficiently large.*

Zhang (1987) proved that Shao's counterexample is the only counterexample, that is, except for $n = 11$ and the number 48, Lewin and Vitek's conjecture is true!

I now turn to the role of digraphs in the theory of the Jordan invariants of matrices.

In their book Turnbull and Aitken (1932) gave a proof for the *existence* of the Jordan canonical form of a square matrix which implicitly used the digraph of a matrix. This connection with digraphs was explicitly made by Brualdi

[11] Maximal subintervals of $\{1, 2, \dots, n^2 - 2n + 2\}$ disjoint from E_n.

(1987b). The basic idea is this. Start with a matrix A of order n. By Jacobi's theorem A is similar to a triangular matrix. Using simultaneous row and column operations, one easily shows that a triangular matrix is similar to a direct sum of triangular matrices each of which has a constant main diagonal, that is, has only one distinct eigenvalue. Hence it is enough to show that a triangular matrix T with all diagonal elements equal to zero (that is, a nilpotent, triangular matrix) has a Jordan canonical form. Now the digraph of the matrix T is very special; it is *acyclic*, which simply means that there aren't any directed cycles. The digraph of a Jordan block is just a path (joining the vertex corresponding to the first row and column to the vertex corresponding to the last row and column). So the Jordan canonical form of T is a matrix similar to T whose digraph consists of one or more vertex-disjoint paths.[12] Let T' be the triangular matrix obtained from T by deleting its last row and column. By induction T' can be assumed to be in Jordan canonical form. Then the digraph of T has a very simple structure; it consists of a number of vertex-disjoint paths and a number of other paths which terminate at the last vertex but are otherwise vertex-disjoint. What remains is to "separate" this latter group of paths. Simultaneous row and column operations can now be used to delete the last arc of all but the longest path of this group. Now the entire collection of paths is pairwise vertex-disjoint, and this means that the matrix is now in Jordan canonical form.

Aitken (1934) and then Littlewood (1936) determined the Jordan invariants of the tensor product of two matrices and those of compound and induced matrices associated with a single matrix. In each case it suffices to assume that the given matrices are themselves in Jordan canonical form. The Jordan canonical form of the constructed matrix (tensor product, compound matrix, induced matrix) can be determined once one knows how to determine it when there is initially only one Jordan block. That is, it suffices to determine the Jordan invariants of the tensor product of two matrices whose digraphs are paths and the Jordan invariants of compound and induced matrices associated with a matrix whose digraph is a path. Aitken made a false combinatorial assumption in his derivation, which was propagated by Littlewood. The resulting error was corrected by Brualdi (1985) for the tensor-product construction, and by Brualdi (1987a) and Brualdi and Chavey (1990) in part for the compound-matrix construction. A careful combinatorial analysis of the digraph of the tensor-product, compound, and induced matrices of Jordan blocks is required. It no longer suffices to assume that the eigenvalue of the

[12] Some paths may have length equal to zero, that is, may contain only one vertex. If the matrix is diagonalizable, all paths have length zero.

Jordan blocks is zero, since the digraph of the constructed matrix depends on whether the eigenvalue is zero or nonzero.

Gansner (1981) and Saks (1986) introduced the combinatorial idea needed to salvage something of the Aitken-Littlewood approach. Let Γ be a digraph of order n. For each nonnegative integer k, let p_k denote the largest cardinality of a set of vertices which can be partitioned into k (possibly empty) sets each of which is the set of vertices of a path.[13] It follows that there is a positive integer s such that

$$0 = p_0 < p_1 < \cdots < p_{s-1} = p_s = n.$$

Now let A be a nilpotent matrix of order n whose nonzero elements are algebraically independent indeterminates. Then the digraph of A does not have any directed cycles, and it follows that the rows and columns of A can be simultaneously permuted to obtain a triangular matrix. Thus without loss of generality A is a triangular matrix. Gansner and Saks show that, in the notation of the above paragraph, where Γ now is the digraph of A, the Jordan invariants of A are

$$p_1 - p_0, p_2 - p_1, \ldots, p_s - p_{s-1}.$$

In particular, the Jordan invariants are combinatorially determined. Since the nonzero elements of the matrix A are generic, no "accidental" cancellations can occur in calculations.[14]

Schneider (1956, 1986) initiated the general study of the Jordan invariants corresponding to the maximal eigenvalue ρ of a (not necessarily irreducible) nonnegative matrix A. By Frobenius' theorem, if A is irreducible, then ρ is a simple eigenvalue, and hence there is only one Jordan block associated with 0 and it has order 1 (thus 1 is the only Jordan invariant associated with 0). If A is reducible, the Jordan structure associated with ρ is closely connected with a certain acyclic digraph or partial order associated with A. First, we may

[13] The *false* assumption that Aitken made was that p_k can be obtained by applying the following greedy algorithm: Choose a longest path γ_1 and remove its vertices from the digraph (and all arcs that touch at least one of the removed vertices), choose a longest path γ_2 from the remaining digraph and remove its vertices, ..., choose a longest path γ_k from the remaining digraph and remove its vertices. Then p_k equals the total number of removed vertices.

[14] There were two flaws in the Aitken-Littlewood approach. One was not to properly define the p_k's and then calculate them for the various matrix constructions considered. The other was not to worry more about possible cancellation of terms in their computations.

assume that A is in *Frobenius normal form*:

$$A = \begin{bmatrix} A_{11} & O & \cdots & O \\ A_{21} & A_{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix},$$

where the diagonal blocks $A_{11}, A_{22}, \ldots, A_{pp}$ are irreducible. Thus the digraph of A has p strong components, one corresponding to each of the diagonal blocks. Define a partial order (or transitive acyclic digraph) on the strong components, more precisely on the set $\{1, 2, \ldots, p\}$, by $i \preceq j$ if and only if $i = j$ or there is a vertex x in the ith strong component and a vertex y in the jth strong component such that there is a path from y to x. Let $\mathscr{S}(A)$ be the subposet induced on those elements $i \in \{1, 2, \ldots, p\}$ such that ρ is an eigenvalue of the irreducible component A_{ii} of A.[15] The following two results were obtained by Schneider (1956, 1986):

Each of the Jordan blocks of A corresponding to the eigenvalue ρ is of order 1 if and only if $\mathscr{S}(A)$ is totally unordered.

There is only one Jordan block corresponding to the eigenvalue ρ if and only if $\mathscr{S}(A)$ is totally ordered.

In his investigation of the generalized eigenspace corresponding to the eigenvalue ρ of the matrix A, Rothblum (1975) obtained a more general combinatorial characterization of the *index* of ρ, that is, the order of the largest Jordan block of A with eigenvalue ρ:

The index of the eigenvalue ρ equals the length of the longest chain in the poset $\mathscr{S}(A)$.

The further relationship between the Jordan structure corresponding to the eigenvalue ρ of the nonnegative matrix A and the poset $\mathscr{S}(A)$ has been subsequently investigated by several authors, including Richman and Schneider (1978), Schneider (1986), Friedland and Hershkowitz (1988), Hershkowitz and Schneider (1989), and Hershkowitz, Rothblum, and Schneider (1989) (see these papers for other references, including papers that have not yet been published).

[15] $\mathscr{S}(A)$ is usually called the *singular (di)graph* of A. This is because if we replace A by the corresponding M-matrix $B = \rho I - A$, then the eigenvalue ρ "turns into" 0. Thus the vertices of $\mathscr{S}(A)$ correspond to the singular blocks in the Frobenius normal form of B.

I now show how the digraph associated with a complex matrix $A = [a_{ij}]$ of order n can give information about the entire spectrum of A. This connection was hinted at in Example 2 of Section 1. The most famous inclusion region for the eigenvalues of A is that of Geršgorin (1931). Let

$$r_k = \sum_{j \neq k} |a_{kj}| \qquad (k = 1, 2, \ldots, n)$$

be the sum of the moduli of the off-diagonal elements in row k of A. Then:

Each eigenvalue of A lies in the union of the n disks

$$D_k = \{z : |z - a_{kk}| \leqslant r_k\} \qquad (k = 1, 2, \ldots, n)$$

of the complex plane.

It was Taussky (1949) who first observed that the combinatorial structure of A provided some additional information about the nature of the eigenvalues of A:

If the matrix A is irreducible (that is, the digraph associated with A is strongly connected), then a boundary point λ of the Geršgorin region $\bigcup_{k=1}^{n} D_k$ can be an eigenvalue of A only if λ is a boundary point of each of the disks D_k.

Brualdi (1982) showed that the digraph could be further used to refine the Geršgorin inclusion region for the eigenvalues.

Let A be an irreducible matrix. For each directed cycle γ, of length greater than one, of the digraph of A, let a region in the complex plane be defined by

$$D(\gamma) = \left\{z : \prod_{\gamma} |z - a_{jj}| \leqslant \prod_{\gamma} r_j\right\},$$

where the products are over all vertices j which belong to γ. Then each eigenvalue lies in the union

$$\bigcup_{\gamma} D(\gamma)$$

over all directed cycles γ of length greater than 1. A boundary point λ of this region can be an eigenvalue of A only if λ is a boundary point of each of the $D(\gamma)$.

The Geršgorin inclusion region was generalized by Feingold and Varga (1962) and Fiedler and Pták (1962) to partitioned matrices by using norms.[16] The more general inclusion region above can be similarly generalized. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix}$$

be a partitioned matrix where the diagonal block A_{ii} is a square matrix of order n_i $(i = 1, 2, \ldots, s)$. Let $\|\cdot\|_i$ be any norm on the complex space $\mathscr{C}^{n_i}$ $(i = 1, 2, \ldots, s)$. These norms induce matrix norms on n_i-by-n_j matrices, and we use the same notation $\|\cdot\|$ for each of these induced norms:

$$\|A_{ij}\| = \sup\left\{ \frac{\|A_{ij}x\|_i}{\|x\|_j} : 0 \neq x \in \mathscr{C}^{n_j} \right\}.$$

We have

$$\begin{aligned} \|A_{ii}\|' := \inf_{x \neq 0} \frac{\|A_{ii}x\|_i}{\|x\|_i} \\ = \begin{cases} \|A_{ii}^{-1}\|_i^{-1} & \text{if } A_{ii} \text{ is invertible,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So we define

$$\|A_{ii}^{-1}\|^{-1} = 0 \qquad \text{if } A_{ii} \text{ is singular.}$$

Let

$$B = \begin{bmatrix} \|A_{11}^{-1}\|^{-1} & \|A_{12}\| & \cdots & \|A_{1s}\| \\ \|A_{12}\| & \|A_{22}^{-1}\|^{-1} & \cdots & \|A_{2s}\| \\ \vdots & \vdots & \ddots & \vdots \\ \|A_{s1}\| & \|A_{s2}\| & \cdots & \|A_{ss}^{-1}\|^{-1} \end{bmatrix},$$

and let

$$T_k = \sum_{j \neq k} \|A_{kj}\| \qquad (k = 1, 2, \ldots, s).$$

[16] Even more general results appear in Fiedler (1961) and Ostrowski (1961).

We then have:

Assume that the matrix B is irreducible. Then each eigenvalue λ of the partitioned matrix A satisfies

$$\prod_{j\in\gamma}\left\|\left(\lambda I_j - A_{jj}\right)^{-1}\right\|^{-1} \leqslant \prod_{j\in\gamma} T_j$$

for at least one directed cycle γ of the digraph of the matrix B.

3. EARLY USE OF MATRIX THEORY IN SOLVING COMBINATORIAL PROBLEMS

There has been a substantial history (which continues strongly today) of the use of matrix theory as a tool in proving combinatorial theorems.[17] One of my favorite (and quite impressive) instances of this is the theorem of Tutte which gives necessary and sufficient conditions for a graph[18] to have a 1-factor. As for bipartite graphs, a *1-factor* of a graph G of order n is a set of edges which touch each vertex exactly once. In order for G to have a 1-factor it is clearly necessary that the number n of vertices be even. If G is a bipartite graph, then König (1931, 1936) discovered necessary and sufficient conditions for G to have a 1-factor and more generally characterized the largest number of edges in G no two of which touch:

Let G be a bipartite graph. Then the largest number of edges of G that do not touch equals the smallest number of vertices that touch all edges. In particular, if G is a spanning subgraph of the complete bipartite graph $K_{m,m}$, then G has a 1-factor if and only if m is the smallest number of vertices that touch all edges.[19]

[17] Or what appear from their statements to be combinatorial theorems. This is more evidence of the view that a matrix is a weighted combinatorial object.

[18] The usual convention is that in a *graph* there are no loops, that is, edges joining a vertex to itself. This is what we shall mean by a graph unless we explicitly mention that loops are permitted.

[19] Hall (1935) formulated and proved an equivalent theorem in the language of families of sets $S_1, S_2, \ldots, S_m$: There exist distinct elements $e_1, e_2, \ldots, e_m$ such that $e_i \in S_i$ ($i = 1, 2, \ldots, m$), if and only if $|\bigcup_{i\in I} S_i| \geqslant |I|$ for all subsets I of $\{1, 2, \ldots, m\}$. More about the connection between sets and bipartite graphs (matrices!) later.

In terms of matrices, König's theorem is:

The largest number of nonzero elements of a matrix A with no two on the same row or column equals the minimum number of rows and columns that contain all of the nonzero elements of A.

It was by no means obvious at the time how to extend König's theorem to nonbipartite graphs. It was a key insight of Tutte (1947) that there are simply stated necessary and sufficient conditions for a graph to have a 1-factor:

A graph G has a 1-factor if and only if for each subset S of its vertices, the number p(S) of connected components with an odd number of vertices of the graph obtained from G by removing the vertices in S (and all the edges that these vertices touch) does not exceed $|S|$:

$$p(S) \leqslant |S|.$$

This was a tremendous discovery at the time. Now there are purely combinatorial proofs of this theorem and it can be made to follow from König's theorem, but Tutte's original proof was matrix-theoretic. The basic ingredients of Tutte's original proof are the following:

(1) Let A be the adjacency matrix of G (the matrix which gives each edge of G a weight of 1).

(2) Let $Z = [z_{ij}]$ be a skew-symmetric matrix of algebraically independent indeterminates (thus the elements above the main diagonal of Z are independent indeterminates, the elements on the main diagonal equal 0, and the elements below the main diagonal satisfy $z_{ji} = -z_{ij}$).

(3) Let $M = [m_{ij}] = A * Z = [a_{ij} z_{ij}]$, the *Hadamard product* of A and Z. Then, as shown by Cayley (1854),

$$\det M = (\text{Pfaffian } M)^2.$$

The classical definition of the Pfaffian is equivalent to the Pfaffian being the sum of the weights of the 1-factors of G.[20]

[20] More precisely, let $\{i, j\}$ be an edge, where we write edges in such a way that $i < j$, and let the weight of $\{i, j\}$ be m_{ij}. Let $\{i_1, i_2\}, \{i_3, i_4\}, \ldots, \{i_{n-1}, i_n\}$ be a 1-factor where $i_1 < i_3 < \cdots < i_{n-1}$. Let π be the permutation $(i_1, i_2, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$. The weight of the 1-factor is then defined to be the product of the sign of π and the weights of its edges. Kastelyn (1961) gave a wonderful combinatorial proof of Cayley's result. It's a combinatorial identity!

(4) Because the elements above the main diagonal of Z are independent indeterminates, we have Pfaffian $M \neq 0$ if and only if G has a 1-factor. By Cayley's identity, det $M \neq 0$ if and only if G has a 1-factor.

(5) Tutte's condition given above is (upon reflection) a necessary condition for G to have a 1-factor. Jacobi's theorem[21] is then used to show that if G does not have a perfect matching, that is, if Pfaffian $M =$ det $\mathrm{M} = 0$, then Tutte's condition is violated.

König's theorem can be derived directly from Tutte's theorem. Kung (1984) gave a direct proof of König's theorem which also uses indeterminates and Jacobi's theorem.

The proof of Tutte outlined above contains one of the early uses of indeterminates in proving theorems in combinatorics. One of the first uses of indeterminates in characterizations of combinatorial concepts is due to Frobenius (1912, 1917). A square matrix is called *fully indecomposable* provided it is not possible to permute its rows and columns independently to obtain a matrix of the form

$$\begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix}$$

where A_1 and A_2 are square matrices.

Let A be a square matrix whose nonzero elements are algebraically independent indeterminates. Then A is a fully indecomposable matrix if and only if det A *is an irreducible polynomial in the indeterminates of* A.

Ryser (1973) rediscovered Frobenius' characterization theorem. Both Ryser (1975) and Schneider (1977) gave algebraic characterizations of irreducible matrices.

One of the earliest uses of matrix theory in combinatorics occurs in the work of Kirchoff (1847), in which he gives a determinant formula for the number of spanning trees of a graph. Let G be a graph of order n, and let A be its adjacency matrix. Let D be the diagonal matrix whose ith diagonal element equals the degree of the ith vertex, that is, the number of edges touching it. The matrix

$$L = D - A$$

is called the *Laplacian matrix*[22] and is a positive semidefinite matrix with at least one eigenvalue equal to 0.

[21] The theorem that shows how to express the determinant of a submatrix of the adjugate or inverse of a matrix A in terms of the determinant of the complementary submatrix of A.

[22] It corresponds to a discrete Laplacian operator.

The absolute value of the determinant of each *submatrix of L of order $n - 1$ equals the number of spanning trees of G. In particular, G is connected if and only if the rank of L equals $n - 1$—equivalently, 0 is a simple eigenvalue of L.*

A consequence of this result is Cayley's formula n^{n-2} for the number of trees with vertices $\{1, 2, \ldots, n\}$.

Kirchoff's theorem was extended to digraphs by Borchardt (1860) and independently by Tutte (1948), and later to weighted digraphs. The matrix A is now an arbitrary matrix of order n (a weighted digraph of order n), and D is the diagonal matrix of row sums.[23]

The determinant of the principal submatrix of L obtained by deleting row and column k equals the sum of the weights of the spanning arborescences[24] *rooted at vertex k.*

Combinatorial proofs of this result have been given by Orlin (1978), Chaiken (1982), and Zeilberger (1985).

Let A denote again the adjacency matrix of a graph G of order n, and let $L = D - A$ denote the Laplacian matrix. As already observed, the matrix L is positive semidefinite and the graph G is connected if and only if 0 is a simple eigenvalue of L. Let $\mu(G) = \mu(L)$ denote the second smallest eigenvalue of L. Then $\mu(G) \geqslant 0$ with equality if and only if G is not connected. Fiedler (1973, 1975, 1990) proposed $\mu(G)$ as an algebraic measure of the connectivity[25] of G and derived many remarkable results.

4. COMBINATORIAL MATRIX IDENTITIES

One of the most fundamental identities for matrices is that identity known as the *Cayley-Hamilton theorem*. Let A be a matrix of order n, and let

$$\chi(x) = \det(xI_n - A) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_{n-1}x + \sigma_n$$

[23] So if A is the adjacency matrix of a digraph, then D is the diagonal matrix of outdegrees of the vertices.

[24] A *spanning arborescence* rooted at vertex k of a digraph is a spanning subdigraph which has no directed cycles for which each vertex other than k has outdegree equal to 1.

[25] In contrast to the combinatorial measures of connectivity given by the *vertex connectivity* and *edge connectivity*.

be the characteristic polynomial of A. There are many different proofs for the fact that A satisfies its characteristic polynomial:

$$\chi(A) = O.$$

To me the most interesting is that of Rutherford (1964), who showed (in the language we are using here) that this identity is a combinatorial identity concerning the weights of walks and directed cycles in the weighted digraph associated with the matrix A. This proof was rediscovered by Straubing (1983) and is presented in Zeilberger (1985) and Brualdi (1990).

Let $x_1, x_2, \ldots, x_k$ be indeterminates. The *standard polynomial* of degree k in $x_1, x_2, \ldots, x_k$ is the polynomial

$$[x_1, x_2, \ldots, x_k] = \sum_{\pi} (\operatorname{sign} \pi) x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(k)},$$

where the summation extends over all permutations π of $\{1, 2, \ldots, k\}$. The theorem of Amitsur and Levitzki (1950, 1951) asserts that matrices of order n satisfy the standard polynomial of degree $2n$, that is:

If $A_1, A_2, \ldots, A_{2n}$ are matrices of order n, then

$$[A_1, A_2, \ldots, A_{2n}] = O.$$

Swan (1963, 1969) showed that this too is a combinatorial identity. It is an identity concerning signed Eulerian trails[26] in a certain weighted digraph with n vertices and total arc weight equal to $2n$.[27]

One of the most intriguing identities involving the determinant is the *master theorem for permutations* of MacMahon (1915). This theorem identifies combinatorially the coefficients in an infinite expansion of the inverse of a certain determinant. Let $A = [a_{ij}]$ be a matrix of order n, and let Y be a diagonal matrix of order n whose diagonal elements are n algebraically independent indeterminates.

The coefficient $A(m_1, m_2, \ldots, m_n)$ in the expansion

$$\det(I_n - AY)^{-1} = \sum_{(m_1, m_2, \ldots, m_n)} A(m_1, m_2, \ldots, m_n) y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$$

[26] An *Eulerian trail* in a digraph is a directed walk which includes each arc of the digraph exactly once.

[27] The number $2n$ arises as follows: It is enough to establish the Amitzur-Levitzki identity for the standard basis of the space of matrices of order $2n$. Thus each of the matrices determines one arc in a digraph with vertices $\{1, 2, \ldots, n\}$. The same arc may be determined several times, and this results in the arcs being weighted with positive integers.

equals the coefficient of $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ *in the product*

$$\prod_{i=1}^{n} \left(a_{i1} y_1 + a_{i2} y_2 + \cdots + a_{in} y_n\right)^{m_i}.$$

Foata [see Cartier and Foata (1969) and Zeilberger (1985)] gave a proof of this result which is an elegant instance of combinatorial reasoning. Again the proof can be phrased in terms of weighted digraphs! Other identities of a similar nature involving both the determinant and the permanent with combinatorial verifications have been obtained by Vere-Jones (1984, 1988) and Chu (1985). These identities concern expansions of $\det(I_n - AY)^{-1}$, $\det(A + Y)$, and $\operatorname{per}(A + Y)$.

5. THE INCIDENCE MATRIX

Another important link between combinatorics and matrix theory is provided by the *incidence matrix*[28]. The incidence matrix provides a link between *combinatorial designs* or more general set systems and matrix theory. It came into prominence in the 1950s, especially through the pioneering work of R. C. Bose and H. J. Ryser.

Let $X = \{x_1, x_2, \dots, x_v\}$ be a set of v elements, and let $X_1, X_2, \dots, X_b$ be b (not necessarily distinct) subsets of X. The *incidence matrix* of this configuration is the b-by-v matrix $A = [a_{ij}]$ for which

$$a_{ij} = \begin{cases} 1 & \text{if} \quad x_j \in X_i, \\ 0 & \text{if} \quad x_j \notin X_i. \end{cases}$$

Let $\mathbf{1}$ denote the all-1's column vector of size b or the all-1's column vector of size v. The size of the vector that $\mathbf{1}$ denotes will be clear from the context. Then $A\mathbf{1}$ is a vector whose entries are the sizes of the sets X_i, and $\mathbf{1}^T A$ is a vector whose entries are the sizes of the elements x_j.[29] The entries of the matrix AA^T of order b are the sizes of the intersections $X_i \cap X_j$; in particular the diagonal entries are the set sizes. The entries of the matrix $A^T A$ of order v are the sizes of the "element intersections," that is, the number of the sets which contain both x_i and x_j; in particular the diagonal entries are

[28] The first link is the (weighted) adjacency matrix or the (weighted) bipartite graph, digraph, or graph associated with a matrix. The incidence matrix can be viewed as the adjacency matrix of a bipartite graph, but for purposes of application to different situations it is better to think of it in the terms discussed in this section.

[29] Element size is a "dual" notion to set size. The *size* of the element x_j as used here is the number of the sets $X_1, X_2, \dots, X_b$ which contain it.

the element sizes. We can summarize this as follows:

$A\mathbf{1}$	:	set sizes
$\mathbf{1}^T A$	:	element sizes
AA^T	:	intersection sizes of sets
$A^T A$	:	intersection sizes of elements

The importance of the incidence matrix in combinatorics is now already clear: *A vast amount of combinatorial information is translated into algebraic statements, and the powerful tools of matrix theory can be applied.*

In the theory of *designs* the sets X_i are usually called *blocks*, and we shall use this terminology. A configuration is a 2-*design* provided all the blocks have the same size k and the sizes of the intersections of distinct elements have the same value λ. In a 2-design the element sizes must all be equal to the same number r. Hence the matrix equations expressing the fact that a configuration is a 2-design are

$$A\mathbf{1} = k\mathbf{1} \quad \text{and} \quad A^T A = (r - \lambda) I + \lambda J,$$

where J denotes the all-1's matrix of order v.

One usually imposes the nontriviality assumption on a 2-design that $v > k$; equivalently, that $r > \lambda$. In this case the matrix $(r - \lambda)I + \lambda J$ is easily seen to be a positive definite matrix of order v. Since the existence of a 2-design implies that this matrix can be factored into a v-by-b matrix and a b-by-v matrix, we have

Fisher's inequality. $b \geqslant v$, *that is, in a 2-design there are at least as many blocks as elements.*

This derivation of this important inequality by some elementary matrix theory is impressive for its elegance and simplicity, and already indicates that matrix techniques should have a large impact on the theory of designs. That this is indeed the case can be seen in the books by Ryser (1963), Hall (1986), and Beth, Jungnickel, and Lenz (1986). The theory of combinatorial designs is a well-developed part of combinatorics, and a vast amount of information is known.[30] We shall scarcely touch it. Wilson (1982, 1984) generalized Fisher's inequality and another set of inequalities for 2-designs called Connor's inequalities[31] to higher-order designs called t-designs.[32]

[30] But there are a lot of intriguing open questions.

[31] *Connor's inequalities* are the statements that the determinants of the principal submatrices of a certain positive semidefinite matrix Q are nonnegative. The matrix Q is the orthogonal projection matrix onto the orthogonal complement of the column space of the incidence matrix A.

[32] In a t-design every set of t elements is in the same number of blocks.

A 2-design in which the number b of blocks equals the number v of elements, that is, equality holds in Fisher's inequality, is called a *symmetric design*.[33] By a clever matrix calculation Ryser (1950) proved that the incidence matrix A of a symmetric 2-design is normal. It follows that A^T is also the incidence matrix of a 2-design, the *dual design* obtained by interchanging sets and elements. A *pairwise balanced design* differs from a 2-design in that the block sizes need not be constant. Under the nontriviality assumption that the block sizes are all strictly greater than λ, the incidence matrix A is still positive definite, and it follows that Fisher's inequality holds as well for pairwise balanced designs.[34] Using elegant matrix calculations, Ryser (1968) and Woodall (1970) independently showed:

A pairwise balanced design with $b = v$ either is a symmetric 2-design[35] *or has exactly two block sizes k_1 and k_2 that satisfy $k_1 + k_2 = v + 1$.*

It remains an open problem to characterize how the above pairwise balanced designs with two block sizes arise. It is conjectured that they all can be obtained from 2-designs by a simple complementation rule.

6. EIGENVALUE TECHNIQUES IN GRAPH THEORY

There is a long history of using eigenvalues in graph theory. A graph of order n has an adjacency matrix A of order n. Since A is symmetric, its n eigenvalues are real, and they are called the *eigenvalues of the graph G*. Eigenvalue techniques are heavily used in the theory of *strongly regular graphs*, that is, regular, connected graphs with only three distinct eigenvalues,[36] developed by Bose (1963), Seidel (1968, 1969), and others, and more generally *distance-regular graphs of diameter d*,[37] developed by Biggs (1974), Delsarte (1973), and others. These subjects require more than the small space we have available here, and we refer the interested reader to the recent book by Brouwer, Cohen, and Neumaier (1989).

The well-known *graph isomorphism problem*[38] is still unsolved, in the sense that it is not known whether there is a polynomial algorithm to

[33] This is an unfortunate bit of terminology. The incidence matrix of a symmetric 2-design does not have to be symmetric, only square.

[34] This is usually attributed to Majindar (1962).

[35] That is, all the block sizes are the same.

[36] One of these is the degree of regularity of the graph G, and this is the eigenvalue ρ discussed in Section 2.

[37] A distance-regular graph of diameter $d = 2$ is a strongly regular graph.

[38] The decision problem: Given two graphs of order n, are they isomorphic?

determine whether or not two graphs are isomorphic. It was a short-lived hope that the eigenvalues of connected graphs might be a complete set of invariants for graph isomorphism. But Collatz and Singowitz (1957) exhibited a pair of connected graphs of order 6 which had the same eigenvalues but were not isomorphic.[39] Hoffman (1963) constructed a pair of cospectral, regular bipartite graphs. Schwenk (1973) showed that for almost all trees T there exists a cospectral mate, that is, a tree T' such that T and T' are cospectral.

Our primary interest in this section is with the use of eigenvalues (which are matrix-theoretic invariants for similarity) in estimating graph-theoretic invariants. The value in doing this is clear. Eigenvalues are computable; one can get excellent accuracy from numerical techniques. However, graph-theoretic invariants are often notoriously difficult to compute, and many of them, when phrased as decision problems, are NP-complete or NP-hard. Thus the best one might hope to achieve is an estimate for a graph-theoretic invariant.

Let G be a connected graph of order n, and let $\rho = \rho(G)$ be its maximal eigenvalue. Wilf (1967) showed that:

The chromatic number $\chi(G)$ of G satisfies

$$\chi(G) \leqslant \rho(G) + 1$$

with equality if and only if G is the complete graph of order n or n is odd and G is a cycle.

This inequality is an improvement of an inequality of Brooks (1941) which asserts that

$$\chi(G) \leqslant (\text{maximal degree of a vertex of } G) + 1$$

with the same conditions for equality as above. While Brooks's inequality is easy to compute for a given graph, the presence of one vertex of large degree can result in a very inaccurate estimate for the chromatic number. On the other hand, the maximum eigenvalue is not too susceptible to a few vertices of large degree.

Let $\rho'(G)$ denote the smallest eigenvalue[40] of G. A lower spectral estimate for the chromatic number of G was obtained by Hoffman (1977):

$$\chi(G) \geqslant 1 - \frac{\rho(G)}{\rho'(G)}.$$

[39] Nonisomorphic graphs with the same eigenvalues are called *cospectral*.

[40] The smallest eigenvalue will be negative for any connected graph of order $n > 1$.

The *stability number* $\alpha(G)$ of G equals the largest number of vertices no two of which are adjacent. The stability number of the complementary graph $\bar{G}$ equals the largest order of a complete subgraph of G. Lovász (1979, 1986) obtained the following bound for the stability number:

$$\alpha(G) \leqslant \frac{-n\rho'(G)}{\rho(G) - \rho'(G)}.$$

This inequality has been further generalized by Lovász. In the proof of these inequalities the interlacing inequalities for symmetric matrices are heavily used.

Recently Mohar (1988, 1989) has explored in depth the use of the *Laplacian eigenvalues* of a graph[41] in estimating graph-theoretic parameters such as the diameter and the mean distance.

A new and important use of eigenvalues of graphs is being made in theoretical computer science. Let n and d be positive integers, and let c be a positive real number. Let G be a bipartite graph of order $2n$ which is a spanning subgraph of $K_{n,n}$ with vertex bipartition $\mathscr{Z}_n$ and $\mathscr{Z}_n'$.[42] Assume also that the degree of each vertex of G is at most d. Then G is called an (n, d, c) *expander* provided that for all subsets X of Z_n with $|X| \leqslant n/2$ we have

$$E(X) \geqslant |X| + \frac{c}{n}(n - |X|)|X|. \tag{*}$$

The quantity $E(X)$ equals the number of vertices (in $\mathscr{Z}_n'$) which are adjacent to at least one vertex in X, and $E(X) - |X|$ measures the "expansion" of X. *Expanders* are thus bipartite graphs with good expansion properties as given by (*). The inequality (*) asserts that X expands by an amount jointly proportional to its size $|X|$ and the quantity $n - |X|$ by which it can expand. The proportionality factor c/n depends only on n and not on the subset X. The inequality (*) is usually written as

$$E(X) \geqslant |X| + c\left(1 - \frac{|X|}{n}\right)|X|,$$

with the constant c as the proportionality factor for $|X|$ and $1 - |X|/n$. In a *strong(n, d, c) expander*, the inequality (*) holds for all subsets X of $\mathscr{Z}_n$.

[41] That is, the eigenvalues of the Laplacian matrix of a graph.

[42] Here we are forced to use a different notation for the two halves of the vertex set.

Expanders are used in parallel sorting and in the construction of graphs with not too many edges but with nonetheless high connectivity properties, so-called *superconcentrators*. The problem is to find linear families of (n_i, d, c) expanders for which

$$\lim_{i\to\infty} n_i = \infty \quad \text{and} \quad \lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1.$$

Using probabilistic arguments, it is possible to prove the existence of linear families of strong expanders, but their explicit construction is more difficult. One problem is that it is not easy to check that a bipartite graph has good expansion properties.[43] We refer to Alon (1986) and the references provided therein for more detailed information about expanders and applications.

Alon (1986) has investigated nonbipartite analogues of expanders. A graph G of order n with each vertex of degree at most d is an (n, d, c) *magnifier* provided that for all subsets X of vertices with $|X| \leqslant n/2$

$$E(X) \geqslant c|X|,$$

where the *expansion* $E(X)$ of X is now the number of vertices not in X which are adjacent to at least one vertex in X. As pointed out by Alon, the double cover[44] of an (n, d, c) magnifier is an $(n, d+1, c)$ expander. Alon (1986) has shown that there is a close relationship between expansion properties of a graph G and the second smallest eigenvalue $\mu(G)$ of its Laplacian matrix L. A graph G of order n each of whose vertices has degree at most d is an (n, d, ϵ) *enlarger* provided $\mu(G) \geqslant \epsilon$. We then have:

If G is an (n, d, ϵ) enlarger, then G is an (n, d, c) magnifier with

$$c = \frac{2\epsilon}{d + 2\epsilon}.$$

If G is an (n, d, c) magnifier, then G is an (n, d, ϵ) enlarger with

$$\epsilon = \frac{c^2}{4 + 2c^2}$$

---

[43] Inequality (*) has to be checked for all subsets X of size at most $n/2$ and for all subsets for strong expanders. It has been proved by Blum et al. (1981) that to check that a graph is an $(n, d, 0)$ expander is a coNP-complete problem, that is, the decision problem "Is a graph not an $(n, d, 0)$-expander?" is an NP-complete problem.

[44] The *double cover* of a graph G of order n with vertex set $\mathscr{X}_n$ is the bipartite graph of order $2n$ with vertex bipartition $\mathscr{X}_n$ and $\mathscr{X}_n'$ in which i and j' are adjacent if and only if $i = j$ or i and j are adjacent in G.

A consequence of these two results is the following:

If a graph G is an (n, d, c) expander, then one can prove efficiently (by computing eigenvalues) that G is an (n, d, c′) expander, where

$$c' = \frac{c^2}{c^2 + d(2 + c^2)}.$$

In other words, if one can get a good estimate on the second largest eigenvalue $\mu(G)$ of the Laplacian, then one can get a good estimate on the expansion properties of a graph. Lubotzky, Phillips, and Sarnak (1988) constructed for each fixed prime p an infinite family of regular graphs of degree $d = p + 1$ with second smallest eigenvalue of their Laplacians bounded below by $d - 2\sqrt{d-1}$ and thus with provably good magnifying properties. There are many additional references, but we close this section by mentioning only Bien (1989) (a survey article) and Chung (1989).

7. QUALITATIVE AND STRUCTURAL THEORY OF LINEAR SYSTEMS

The study of sign-solvable linear systems is largely a combinatorial study. Let $A = [a_{ij}]$ be a real matrix of size m by n, and let b be a vector of size m. Then the system

$$Ax = b$$

is *sign-solvable* provided it is solvable and both its solvability and the sign pattern[45] of the solution x depend only on the sign patterns of A and of b. Clearly, in the investigation of sign-solvable systems it suffices to assume that the elements of A and of b are $+1$, -1, or 0.

The study of sign-solvable systems quickly reduces [see Klee, Ladner, and Manber (1984)] to the study of two kinds of matrices, called S-matrices and L-matrices. A p-by-q matrix A' is called an *S-matrix* provided that the columns of each matrix with the same sign pattern as A' are the vertices of a $(q-1)$-simplex in $\mathscr{R}^p$ whose relative interior contains the origin. A matrix A'' is an *L-matrix* provided the columns of each matrix with the same sign pattern as A'' are linearly independent. Let β be the subset of indices j such that in the solution $x = (x_1, x_2, \ldots, x_m)^T$ of $Ax = b$ we have $x_j \neq 0$, and let

[45] The *sign pattern* of a matrix (a vector) is the matrix (the vector) obtained by replacing each of its elements with its sign ($+1$, -1, or 0).

α be the subset of indices i such that $a_{ij} \neq 0$ for some $j \in J$. Let A' be the matrix obtained from A by deleting the columns not in β and replacing column j by its negative whenever $x_j < 0$ ($j \in \beta$), and then appending $-b$ as a final column. Let A'' be the submatrix $A[\bar{\alpha}, \beta]$ of A obtained by deleting the rows with index not in α and the columns with index not in β. Then the system $Ax = b$ is sign-solvable if and only if A' is an S-matrix and A'' is an L-matrix.

S-matrices are basically well understood, and there are polynomial recognition algorithms for them. L-matrices are not yet so well understood, and the recognition problem for m-by-n L-matrices is known to be coNP-complete even when $n = m + \lfloor m^{1/k} \rfloor$ (k is fixed). It is not known whether the recognition problem for square L-matrices has a polynomial algorithm. Because of this much attention has been given to square L-matrices.

A square L-matrix is usually called a *sign-nonsingular matrix*. This is because a square matrix A is an L-matrix if and only if every matrix with the same sign-pattern as A is nonsingular. Since a sign-nonsingular matrix has a nonzero determinant, in investigations concerning sign-nonsingular matrices one may assume without loss of generality that A has all -1's on its main diagonal.[46] It follows easily that the matrix A is sign-nonsingular if and only if each nonzero term in the determinant of A has the same sign, that is, by our convention, has sign equal to $(-1)^n$. From this one gets the following characterization of sign-nonsingular matrices due to Bassett, Maybee, and Quirk (1968):

The matrix A is sign-nonsingular if and only if the weight of each directed cycle[47] *is* 1.

It follows from the above discussion that the matrix A is sign-nonsingular if and only if per $|A| = |\det A|$, where $|A|$ is the matrix obtained from A by replacing each element by its absolute value. But if per $|A| = |\det A|$, then per $X = |\det(A * X)|$ for every matrix X with 0's in positions where A (equivalently $|A|$) has 0's. Thus sign-nonsingular matrices A of order n correspond to conversions of the determinant into the permanent on the coordinate subspaces $\mathscr{M}_n(|A|)$ consisting of all (real or complex) matrices of order n with 0's in at least those positions in which the matrix $|A|$ has 0's.

Little (1975) characterized those matrices E of 0's and 1's for which there exists a sign-nonsingular matrix[48] A with $|A| = E$. His characterization is in

[46] Of course, one can instead assume that A has all 1's on its main diagonal. But there are some good reasons for using -1's.

[47] Recall that the weight of a directed cycle is defined to be the *negative* of the products of the weights of its arcs.

[48] Actually, Little does not discuss sign-nonsingular matrices, only the problem of converting the permanent into the determinant by affixing minus signs to some of its elements.

terms of the bipartite graph associated with A, but can be expressed entirely in matrix terms. A different characterization was obtained by Seymour and Thomassen (1987) in terms of the associated digraph. For more details, the reader may consult Brualdi (1988) and Brualdi and Shader (1990).

The structural or generic approach in systems theory was initiated by Lin (1974) and starting in the early 1980s has been vigorously developed by a number of people, especially Murota (1985, 1987, 1989a, b, 1990), van der Woude (1988, 1990), and Murota and van der Woude (1989). Graph-theoretical and other combinatorial considerations form an integral part of the analysis of such systems. The basic idea is this. Consider a *linear*, *time-invariant dynamical system* (LTIDS)

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

where $x(t)$ is a vector in $\mathscr{R}^n$ representing the *state*, $u(t)$ is a vector in $\mathscr{R}^m$ representing the *input*, and $y(t)$ is a vector in $\mathscr{R}^p$ representing the *output* of the system. The matrices A, B, and C have sizes n-by-n, n-by-m, and p-by-n, respectively. The LTIDS is said to be a *structured system* if the nonzero elements of A, B, and C are algebraically independent indeterminates over $\mathscr{R}$. The structural analysis of such systems is concerned with relationships that are true almost everywhere, that is, outside a proper algebraic variety of $\mathscr{R}^k$, where k is the number of independent parameters.[49]

An LTIDS can be represented by a weighted digraph. For instance, let $m = 2$, $n = 3$, and $p = 2$, and let

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \\ 0 & b_{32} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & 0 \\ 0 & c_{22} & c_{23} \end{bmatrix}.$$

The digraph associated with this system is pictured in Figure 4. The arrows on the left (that is, those leaving the input vertices and entering the state vertices) indicate which inputs affect which states, the arrows in the middle (those leaving and entering state vertices) indicate which states affect other states, and the arrows on the right (those leaving state vertices and entering output vertices) indicate which states affect which outputs. Thus, for instance, the path u_1, x_1, x_3, y_2 from input vertex u_1 to output vertex y_2 implies that the input component u_1 can potentially affect the output component y_2.

[49] Any LTIDS can be analyzed from the structural point of view by treating the nonzero parameters as independent indeterminates. We shall use the modifiers "structural," "structurally," or "generic" when we consider an LTIDS as a structured system.

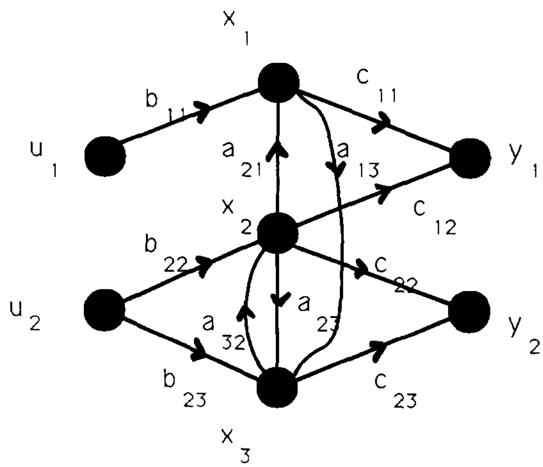

FIG. 4.

The LTIDS is *controllable* provided

$$\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n.$$

The following theorem of Lin(1974), Glover and Silverman (1976), Shields and Pearson (1976), and Maeda (1981) characterizes when a structured system is controllable:

A linear time-invariant dynamical system is structurally controllable if and only if (i) *each state vertex is reachable by a directed walk from at least one input vertex, and* (ii) *the matrix* $[A \mid B]$ *has structural rank equal to* n.[50]

The *transfer matrix* of the LTIDS is the matrix

$$K(x) = C(xI - A)^{-1}B.$$

It was shown by van der Woude (1990) that there is an integer r such that the rank of the transfer matrix $K(x)$ equals r outside a proper algebraic variety of $\mathscr{R}^k$, and this integer r is called the *generic rank* of $K(x)$. Using a Smith-like canonical form for matrices of rational functions of x, one can define the

[50] To say that $[A \mid B]$ has structural rank equal to n means that it has term rank equal to n. This is equivalent to the bipartite graph of $[A \mid B]$ having a matching of size n, which in turn is equivalent to the subdigraph of the digraph associated with the LTIDS determined by the state and input vertices having the property that the vertices can be partitioned into directed cycles and directed paths whose initial vertex is an input vertex.

orders of the zeros at infinity of the transfer matrix.[51] In the case of a structured system van der Woude showed that it is possible to define the *generic* or *structural* orders of the zeros at infinity and proved that:

The generic orders of the zeros at infinity of the structured transfer matrix $K(x)$ *equals* $m_i - m_{i-1}$ $(i = 1, 2, \dots, r)$, *where* $m_0 = 0$ *and* m_i *equals the smallest number of state vertices in any i-tuple of pairwise disjoint paths from the input vertices to the output vertices* $(i = 1, 2, \dots, r)$.

The above discussion is only a brief account of a small part of the recent developments on the structural analysis of dynamical systems. We mention in addition that Murota and Iri (1985) have introduced the notion of *mixed matrices*. These are matrices whose nonzeros are either constants or independent indeterminates and thus whose application to systems theory is perhaps more physically faithful than are matrices all of whose nonzeros are all independent indeterminates. Such matrices have been intensively investigated by Murota (1987, 1989a, b).

8. OTHER TOPICS

I do not mean to imply that the topics to be discussed in this penultimate section are somehow less central or less interesting. Indeed, some of my favorite topics are included here. Nor do I mean to imply that a topic not mentioned at all does not belong to or has no connections with CMT. It is, however, necessary to bring this article to a close, and I shall be even more brief than I have been already.

The mathematical theory of *sparse matrices*, that is, matrices of large size with many zeros whose locations are such that they can be taken advantage of in computations, is largely concerned with combinatorial issues. One of these issues is whether it is possible to choose an ordering for Gaussian elimination which does not introduce nonzero elements in positions which were initially occupied by zeros.[52] Symmetric matrices with such *perfect elimination schemes* are basically equivalent to chordal graphs.[53] These are graphs such that every

[51] The Smith form is a diagonal matrix whose nonzero diagonal elements are x^{-t_1}, $x^{-t_2}, \dots, x^{-t_r}$, where $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r$ and r is the generic rank of $K(x)$. The *orders of the zeros at infinity* are the numbers $t_1, t_2, \dots, t_r$.

[52] This is one example of what is meant by taking advantage of the zeros in computations. A more general issue is to choose an ordering which minimizes or at least keeps relatively small the number of zero positions which become—or might become, depending on the actual numerical quantities—nonzero positions. Positions which are occupied by zeros and which will stay occupied by zeros during a specific computation need not be considered in creating a data structure for the computation.

[53] Also called *triangulated graphs*.

cycle of length at least 4 contains a chord. The corresponding notion for nonsymmetric matrices is that of *chordal bipartite graphs* , which are bipartite graphs for which every cycle of length (necessarily even) greater than 4 has a chord. For more details we refer the reader to Golumbic (1980).

For doing Gaussian elimination on large matrices with parallel processors, one possibility is to "uncouple" the system by deleting some rows and columns (thereby breaking the system up into smaller systems), solve the small systems on separate processors, and then recouple the system to find its solution. The problem is then to find good separators for the uncoupling, and this is largely a graph-theoretic issue [Pothen, Simon, and Liou (1990)].

Chordal graphs arise also in problems concerning the completion of partial matrices.[54] Let G be a graph of order n. Consider the set $\mathscr{C}(G)$ of all partial positive semidefinite Hermitian matrices $A = [a_{ij}]$ of order n with associated graph equal to G.[55] Grone, Johnson, Sá, and Wolkowicz (1984) proved:

Every matrix in $\mathscr{C}(G)$ can be completed to a positive definite Hermitian matrix if and only if G is a chordal graph.

A survey of matrix completion problems is given by Johnson (1990).

Finally we mention that in the study of polytopes of matrices with prescribed row and columns sums, and perhaps a prescribed graph or other prescribed properties (e.g. symmetry), combinatorial considerations arise naturally and are strongly influential. This leads me into integer programming, combinatorial optimization, totally unimodular matrices, etc. I refer the interested reader to the book by Schrijver (1986). Then there are diagonal scalings of matrices, latin squares, Hadamard matrices, and so on.

9. CODA

I hope I have been convincing that matrix theory and combinatorics do enjoy a symbiotic relationship and that this mutually beneficial relationship is so intrinsic that it cannot do other than to continue and to bring rewards for both fields. Unlike most other mathematical fields, matrix theory does not seem to have the "big conjecture" that attracts a lot of attention outside the field.[56] Matrix theory does have its share of elegant theorems, and its applicability to real problems, and to other parts of mathematics, is generally acknowledged. The areas mentioned in this article will, I believe, continue to

[54] Matrices of a specified size not all of whose elements have been specified.

[55] Thus a_{ij} is specified if and only if $\{i, j\}$ is an edge of G, in which case $a_{ji} = \bar{a}_{ij}$ and each fully specified principal submatrix of A has a positive determinant.

[56] One exception might be the van der Waerden conjecture (now theorem) that the minimum permanent of a doubly stochastic matrix of order n is $n!/n^n$.

be active areas of research, and who knows in what directions they might lead us?

I think that one of the greatest strengths of matrix theory is its applicability and usefulness in so many diverse parts of mathematics. *Some* of the best work in matrix theory is done by mathematicians, electrical engineers, statisticians, computer scientists, etc., who don't consider themselves matrix theorists. I believe that for these reasons almost every matrix theorist should be in, or at least closely connected with, at least one other area where matrix theory plays an important role. To isolate ourselves will, in my opinion, lead to an erosion of support within the mathematical-sciences community, and to a doubtful future. I will close with a list of some of the areas that I can think of which have many important connections with matrix theory:

(1) combinatorics and discrete mathematics,
(2) numerical analysis,
(3) systems and control theory,
(4) theoretical computer science,
(5) statistics,
(6) probability,
(7) game theory,
(8) dynamical systems,
(9) mathematical programming,
(10) operations research and network theory,
(11) signal processing,
(12) coding and information theory,
(13) biology and genetics,
(14) ring theory.

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